

Branes and integrability in the N=2 SUSY YM theory

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Abstract

We suggest the brane interpretation of the integrable dynamics behind the exact solution to the N=2 SUSY YM theory. Degrees of freedom of the Calogero type integrable system responsible for the appearance of the spectral Riemann surfaces originate from the collective coordinates of the dynamical branes. The second Whitham type integrable system corresponds to the low energy scattering of branes similar to the scattering of the magnetic monopoles.

1. Since the beautiful derivation of the low energy effective action for N=2 SUSY YM theories [1] it was recognized that the full set of the relevant information about the solution is encoded in the pair of the integrable systems responsible for pure YM theory [2, 3], as well as YM with adjoint [4] or fundamental matter [5]. The solution was formulated in terms of the moduli spaces relevant for the vacuum configuration. Therefore the revealing of the integrable systems can be expected from the very beginning since they typically have different moduli spaces as the configuration or phase spaces. Nevertheless the interpretation of the field theory data in terms of the integrable systems looks as the amusing coincidence and the proper identification of the degrees of freedom in the integrable dynamics has to be found.

In this note we suggest interpretation of the integrable systems in terms of the brane configuration. Namely in IIA picture one has to involve D0-branes and D4-branes which provide the dynamical degrees of freedom as well as NS5 branes which represent the background. It turns out that Toda or Calogero dynamics comes from the D0 and D4 brane degrees of freedom while the Whitham dynamics is related to the slow evolution of D4 branes along the Coulomb branch of the moduli space. We also briefly discuss the corresponding IIB/F theory picture.

2. To get the identification of the degrees of freedom recall the equation of the vacuum curve for the pure SU(N) YM theory.

$$z + \frac{\Lambda}{z} = P_n(x), \quad (1)$$

where P_n is the n-th order polynomial. Variable z is defined on the spectral curve - n-sheet cover of the sphere with the pair of the marked points. The base sphere can be considered as the degeneration of the torus with a marked point which is the bare spectral curve for the elliptic Calogero model. Remind that the modular parameter of the torus is the bare coupling constant in the N=4 SYM theory.

In IIA picture perturbatively one deals with the pair of NS5 branes with worldvolume (x_0, \dots, x_5) and N_c D4 branes with worldvolume (x_0, \dots, x_3, x_6) stretched between them [6] (see also [7]). The distance between NS5 branes in x_6 direction is $\frac{1}{g^2}$. Perturbatively, coordinates of D4 branes in $x_4 + ix_5$ provide the momenta in the integrable system. However to introduce the coordinates in the dynamical system additional N_c D0 branes located at D4 branes

one per each have to be added. These D0 branes represent nonperturbative effects in the field theory. In the M theory these degrees of freedom correspond to the KK excitations on M5 brane wrapped around the spectral curve.

In IIB/F picture we consider N_c parallel 7-branes wrapped around bare torus (or sphere in the Toda case) resulting in the $SL(N, \mathbb{C})$ theory on their world volume [8]. It was argued [9] that 7-brane coordinates normal to the torus are twisted so the $(1,0)$ form Φ on the surface Σ fixes the spectral curve via equation $\det(\Phi(z) - x) = 0$. It can be identified with the Lax operators for the Toda or Calogero systems. Having in mind the relation to the positions in the normal direction we can identify its diagonal elements, playing the role of the momenta of N_c particles p_i as the positions of 7-branes while the nondiagonal elements are due to the strings stretched between the neighbour 5-branes in the Toda case and between all in the Calogero case. The antiholomorphic $(0,1)$ form \bar{A} representing the nonperturbative effects in the field theory is defined on Σ is diagonal $SL(N, \mathbb{C})$ matrix whose entries x_i denote the coordinates of the particles in the dynamical system.

The Wilson line in the proper representation is inserted at the marked point on the torus and appears in the r.h.s of the momentum map equation [10]

$$\bar{\partial}\phi + [\bar{A}, \phi] = M(1 - \delta_{ij})\delta(z). \quad (2)$$

This equation concerns the $(1,0)$ form Φ and $(0,1)$ form \bar{A} defined on the elliptic curve with the marked point. The pair (Φ, \bar{A}) with the momentum map restriction provides the phase space for the elliptic Calogero model. The parameter M which fixes the monodromy around the marked point corresponds to the mass of the adjoint hypermultiplet in the 4d field theory.

Resolution of the momentum map equation results in the expression for the Calogero Lax operator

$$L_{ij}(Z) = \Phi_{ij}(z) = \delta_{ij}p_j + iM(1 - \delta_{ij})\exp\left(\frac{(x_i - x_j)(\lambda - \bar{\lambda})}{\tau - \bar{\tau}}\right)\frac{\sigma(x_i - x_j - z)}{\sigma(z)\sigma(x_i - x_j)}. \quad (3)$$

The position of the Wilson line on Σ represents the common coordinate of the N_c background branes on the torus and its diagonal elements define the coordinates of the background branes along the direction of the Wilson line.

3. Let us find now the interpretation of the equation of motion in the Toda case. The D0 branes on D4 branes in IIA picture behave as monopoles. These D0 branes located on some D4 branes are connected with the other D4 branes by the open strings. Following [11] we can identify the world volume theory of strings as coming from 10d SYM theory via the dimensional reduction. Moreover there is one to one correspondence between the equation defining the ground state in the SUSY σ model and the Nahm equations describing the monopole moduli space. Finally to get the Toda equation in the brane terms we use the observation [12] that the spectral curve for $SL(N, \mathbb{C})$ Toda system coincides with the one for the cyclic N-monopoles. The spectral curve for the cyclic N-monopole follows from the generic spectral curve

$$\eta^n + \eta^{n-1}a_1(\kappa) + \dots + a_n(\kappa) = 0 \quad (4)$$

if one assumes that the center of mass is fixed at the origin, total phase is unity and the symmetry under the cyclic group of order N is imposed.

These arguments result in the following expression for the Toda Lax operator in terms of the Nahm matrixes T_i

$$T = T_1 + iT_2 - 2iT_3\rho + (T_1 - iT_2)\rho^2 \quad (5)$$

$$T_1 = \frac{i}{2} \sum_{j=1} q_j (E_{+j} + E_{-j})$$

$$T_2 = - \sum_{j=1} q_j (E_{+j} - E_{-j}) \quad (6)$$

$$T_3 = \frac{i}{2} \sum_j p_j H_j,$$

where E and H are the standard $SU(N)$ generators, p_i, q_i represent the Toda phase space, and ρ is the coordinate on the CP^1 above. This CP^1 is involved in the twistor construction for monopoles and a point on CP^1 defines the complex structure on the monopole moduli space. With these definitions Toda equation of motion and Nahm equation acquire the simple form

$$\frac{dT}{dt} = [T, A] \quad (7)$$

with the fixed A. Note that to define A operator it is necessary to introduce the fourth Nahm operator T_0 .

To get the additional insight on the nature of the variables it is useful to consider the fermions in the external field. Namely let us consider 3d Dirac operator in the external field

$$(\sigma D + \phi - t)\Psi = 0, \quad (8)$$

which has k linearly independent solutions for the magnetic charge k . That is just the variable t becomes the time variable in the Toda dynamics. The Nahm matrixes come from the fermions in a wellknown way

$$\begin{aligned} T_j &= \int x_j \Psi^+ \Psi d^3x; j = 1, 2, 3 \\ T_4 &= i \int \Psi^+ \frac{d\Psi}{dt} d^3x. \end{aligned} \quad (9)$$

Both expressions have the form of the Berry connection and the momentum space serves as the space of parameters. In the inverse construction one can derive the world sheet monopole fields starting with the auxiliary fermions in the background Nahm connection. The procedure looks as follows;

$$(i \frac{d}{dt} + i\sigma x + \sigma T^+)v = 0 \quad (10)$$

$$\begin{aligned} A_j &= \int v^+ \partial_j v dt \\ \phi &= \int t v^+ v dt, \end{aligned} \quad (11)$$

where the integration intervals are defined through the asymptotic values of the Higgs field. Fermions taken at $x=0$ look like the fermions in the auxiliary spectral problem for the Toda system. Therefore the analogy with the Peierls model discussed in [13] can be partially clarified. Namely equation for the fermions in the Nahm connection can be treated as the Schrödinger equation while the 1d "crystal" with N_c sites is formed by the D0 branes. Cyclic symmetry of the monopole configuration implies the periodicity of the system so the quasimomentum can be introduced for the fermions. Spectral curve of the Toda system becomes the dispersion law for the fermions and the field theory BPS state gets interpreted as the completely filled band. Integrals of motion in the Toda system parametrize the moduli space of the strongly centered cyclic monopoles and can be expressed in terms of the vacuum expectation values of the scalar field in the YM theory.

From the brane picture we get some insight on the second Lax representation for the Toda system by 2×2 matrixes. Remind that Toda transfer matrix for the $SU(N_c)$ case has the following form

$$T = \prod_{i=1}^{N_c} L_i(\lambda - \lambda_i) \quad (12)$$

$$L_i = \begin{pmatrix} p_i - \lambda & \exp(\phi_i) \\ \exp(-\phi_i) & 0 \end{pmatrix}$$

therefore the momenta define the positions of D0 branes in $x_4 + ix_5$ direction while nondiagonal elements are in the correspondence with the instanton-like fermion transitions between the nearest sites. This picture is suggestive for the generalization to the theory with the fundamental matter governed by the spin chains [5]. Indeed for this case one keeps the size of the local L operator assuming the elements of the matrix to be $SL(2, \mathbb{R})$ valued.

4. To recognize the second Whitham type integrable system let us adopt slightly different perspective from the F-theory on the elliptically fibered K3 which is equivalent to the orientifold of type IIB theory or, after T duality, to type I theory on T^2 . Due to [14] we can treat the $N=2$ $d=4$ theory as a world volume theory of probe 3-branes in the background of the splitted orientifold 7 planes placed at points $\pm\Lambda$ in the $u = Tr\phi^2$ complex plane for the $SU(2)$ case. We assume that the possible masses of the fundamental matter tend to infinity so we work with the pure YM case. Note that actually we have $SL(2, \mathbb{C})$ bundle on the elliptic fiber in the F theory, which is the torus with a marked point. Degeneration of the fiber to the sphere corresponding to the pure YM theory is performed in a way providing the emergence of the dimensional transmutation parameter Λ .

The key point is that now we have to consider the dynamics of the 3-branes in the directions transverse to the background 7 branes. The arising dynamics is quite transparent already in the $SU(2)$ case. Let us recall that Whitham dynamics for $SU(2)$ case is governed by the solution of the first Gurevich-Pitaevskii problem [2] which can be easily interpreted as follows. At the initial moment of evolution 3-branes coincide with the one of the orientifold planes and with the other planes at the end of the evolution. Therefore the exact metric on the moduli space can be derived from the 3-brane exchange between the orientifold planes which extends the closed

string exchange picture [15] and gives rise to the perturbative part of the metric on the moduli space. To analyze the Whitham dynamics in SU(2) case it is convenient to use the following form of the spectral curve

$$y^2 = (x^2 - \Lambda^2)(x - u). \quad (13)$$

The point u represents the position of two 3-branes (which are at $\pm\sqrt{u}$ at ϕ plane) and "nonperturbative" branching points provide the fixed positions of the background branes. The branching point u and therefore a pair of 3-branes moves from $u = \Lambda$ to $u = -\Lambda$ according to the Whitham dynamics for the one-gap KdV solution which corresponds to the Seiberg-Witten solution.

$$\eta(x, t) = 2dn^2 \left[\frac{1}{\sqrt{6}} \left(x - \frac{1+s^2}{3} t, s \right) \right] - (1 - s^2) \quad (14)$$

where

$$\frac{1+s^2}{3} - \frac{2s^2(1-s^2)K(s)}{3(E(s) - (1-s^2)K(s))} = \frac{x}{t}, \quad (15)$$

$K(s)$ and $E(s)$ are the elliptic moduli and $s^2 = \frac{u+\Lambda}{2\Lambda}$. In terms of the automodel variable $\theta = \frac{x}{t}$ the left background brane corresponds to $\theta = -1$ while the right one to $\theta = \frac{2}{3}$. Quasiclassical tau-function of this solution provides the prepotential for the SU(2) theory $\mathcal{F} = \log \tau_{qcl}$ [2].

Let us emphasize the close analogy of the motion above with the monopole scattering in the low energy limit. It is known that there are several types of the geodesic motion on the monopole moduli space which governs the low-energy monopole scattering [16]. Among them there is a geodesic motion which results in the transformation of the initial monopoles into the dyons. Let us compare it with the Whitham dynamics under consideration. In the ϕ plane there are two dynamical 3-branes placed at $\pm\phi$ and four background branes placed at $\pm\sqrt{\Lambda}; \pm i\sqrt{\Lambda}$. At the initial moment one has the massless monopoles in the spectrum since the dynamical branes are placed at $\pm\sqrt{\Lambda}$. After the right angle scattering at $u=0$ 3-branes acquire the electric charges. Indeed we know that at the final point of the Whitham evolution $u = -\Lambda$ one has the massless dyons which can be considered as the (1,1) string stretched between the 3-brane and background brane. Therefore the string can end on the 3-brane only if it has the electric charge.

5. To conclude we have shown that degrees of freedom in the finite-dimensional integrable

systems relevant for SYM theories allow the interpretation in terms of branes. Collective coordinates of branes cover the phase space of the integrable system so the question if the quantization of the integrable system is meaningful gets the positive answer. It would correspond to the collective coordinate quantization which provides the proper wave function. Moreover the interpretation of the Whitham dynamics in terms of the brane scattering implies that a kind of "quantization" can provide a description of the "quantum scattering". It would be also important to recognize the direct relation between the monopole moduli spaces considered in this note and the ones appeared in the treatment of the SUSY 3d theories [17]. One more interesting issue to be discussed is the brane interpretation of the relativistic integrable system found in 5d theory [18].

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